

# Twist maps for non-standard quantum algebras and discrete Schrödinger symmetries

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## Abstract

The minimal twist map introduced by Abdesselam *et al* [1] for the non-standard (Jordanian) quantum  $sl(2, \mathbb{R})$  algebra is used to construct the twist maps for two different non-standard quantum deformations of the  $(1+1)$  Schrödinger algebra. Such deformations are, respectively, the symmetry algebras of a space and a time uniform lattice discretization of the  $(1+1)$  free Schrödinger equation. It is shown that the corresponding twist maps connect the usual Lie symmetry approach to these discrete equations with non-standard quantum deformations. This relationship leads to a clear interpretation of the deformation parameter as the step of the uniform (space or time) lattice.

# 1 Introduction

The non-standard quantum deformation of the  $sl(2, \mathbb{R})$  algebra (also known as Jordanian or  $h$ -deformation) [2, 3, 4, 5, 6, 7, 8] has induced the construction of several non-standard quantum algebras which are naturally related with  $sl(2, \mathbb{R})$  by means of either a central extension ( $gl(2)$ ) or contraction (the  $(1+1)$ -dimensional Poincaré algebra  $\mathcal{P}$ ). The contraction of the quantum  $gl(2)$  algebra leads to a quantum harmonic oscillator  $h_4$  algebra which, in turn, can be interpreted as a central extension of the Poincaré algebra  $\mathcal{P}$ . These relationships have been studied in [9] within the context of boson representations and are displayed in the l.h.s. of the following diagram:

$$\begin{array}{ccccccc}
 U_z(sl(2, \mathbb{R})) & \xrightarrow{\text{central extension}} & U_z(gl(2)) & \xrightarrow{\text{Hopf subalgebra}} & U_\tau(\mathcal{S}) & \longrightarrow & \text{Discrete time SE} \\
 \downarrow \varepsilon \rightarrow 0 & & \downarrow \varepsilon \rightarrow 0 & & & & \\
 U_z(\mathcal{P}) & \xrightarrow{\text{central extension}} & U_z(h_4) & \xrightarrow{\text{Hopf subalgebra}} & U_\sigma(\mathcal{S}) & \longrightarrow & \text{Discrete space SE}
 \end{array}$$

The cornerstone of the above four quantum algebras is the triangular Hopf algebra with generators  $J_3, J_+$  verifying  $[J_3, J_+] = 2J_+$ , and with classical  $r$ -matrix [10, 11]

$$r = zJ_3 \wedge J_+. \quad (1.1)$$

The deformed commutator, coproduct and universal quantum  $R$ -matrix are:

$$[J_3, J_+] = \frac{e^{2zJ_+} - 1}{z} \quad \Delta(J_+) = 1 \otimes J_+ + J_+ \otimes 1 \quad \Delta(J_3) = 1 \otimes J_3 + J_3 \otimes e^{2zJ_+} \quad (1.2)$$

$$\mathcal{R} = \exp\{-zJ_+ \otimes J_3\} \exp\{zJ_3 \otimes J_+\}. \quad (1.3)$$

This structure is a Hopf (Borel) subalgebra of all the quantum algebras that we have previously mentioned. We recall that the Jordanian twisting element for the Borel algebra was given in [6], the expression for the  $R$ -matrix (1.3) was deduced in [8] and another construction for  $\mathcal{R}$  can be found in [7]. The quantum algebra (1.2) underlies the approach to physics at the Planck scale introduced in [12, 13].

The triangular nature of a quantum deformation (like (1.2)) ensures the existence of a twist operator (as the one given in [6]) which relates the (classical) cocommutative coproduct with the (deformed) non-cocommutative one [11]. This means that there should exist an invertible twist map which turns the deformed commutation rules into usual Lie commutators. In this respect, a class of twist maps for the non-standard quantum  $sl(2, \mathbb{R})$  algebra has been explicitly constructed by Abdesselam *et al* [1]; amongst these maps we will consider the simplest one, which is called the ‘minimal twist map’. A first aim of this paper is to implement the minimal twist map in the quantum algebras  $U_z(gl(2))$ ,  $U_z(\mathcal{P})$  and  $U_z(h_4)$ , showing their relationships with  $U_z(sl(2, \mathbb{R}))$  through either contraction or central extension; this task is carried out in the section 2. In relation with this kind of twists, we recall that a different deformation map for  $U_z(\mathcal{P})$  was introduced in [14] and a similar construction for

$U_z(e(3))$  was given in [15]. Jordanian twists for  $sl(n)$  and for some inhomogeneous Lie algebras have been also studied in [16], and a general construction of a chain of Jordanian twists has been introduced in [17] and applied to the semisimple Lie algebras of the Cartan series  $A_n$ ,  $B_n$  and  $D_n$ .

On the other hand,  $gl(2)$  and  $h_4$  are Lie subalgebras of the centrally extended  $(1+1)$ -dimensional Schrödinger algebra  $\mathcal{S}$  [18, 19]. These Lie subalgebra embeddings have been implemented at a quantum algebra level and two non-standard quantum Schrödinger algebras have been obtained from them by imposing that either  $U_z(gl(2))$  or  $U_z(h_4)$  remains as a Hopf subalgebra. The former [20],  $U_\tau(\mathcal{S}) \supset U_z(gl(2))$ , has been shown to be the symmetry algebra of a time discretization of the heat or (time imaginary) Schrödinger equation (SE) on a uniform lattice. Likewise, as we shall show in this paper, the latter [21],  $U_\sigma(\mathcal{S}) \supset U_z(h_4)$ , can be related with a space discretization of the SE also on a uniform lattice. These connections are displayed in the r.h.s. of the above diagram. Obviously, when the deformation parameters  $z, \tau, \sigma$  go to zero we recover the usual Lie algebra picture and the continuous SE.

In this context of discrete symmetries, we recall that quantum algebras have been connected with different versions of spacetime lattices through several algebraic constructions that have no direct relationship with the usual Lie symmetry theory [22, 23, 24]. Recent works [25] have also developed new techniques for dealing with the symmetries of difference or differential-difference equations and have tried to adapt in this field the standard methods that have been so successful when applied to differential equations. An exhaustive study for the discretization on  $q$ -lattices of classical linear differential equations has shown that their symmetries obeyed to  $q$ -deformed commutation relations with respect to the Lie algebra structure of the continuous symmetries [26, 27, 28]. However, Hopf algebra structures underlying these  $q$ -symmetry algebras have been not found. When the discretization of linear equations is made on uniform lattices it seems that the relevant symmetries preserve the Lie algebra structure [29, 30]. Perhaps, this is the reason why the symmetry approach to these equations has not been directly related to quantum algebras.

The second and main objective of this paper is to relate the discrete SE's and their associated differential-difference symmetry operators provided by the quantum Schrödinger algebras  $U_\sigma(\mathcal{S})$ ,  $U_\tau(\mathcal{S})$  with the results concerning a discrete SE and operators obtained in [29] by following a Lie symmetry approach. Since the latter operators close the Schrödinger Lie algebra  $\mathcal{S}$ , the crucial point in our procedure is to find out the twist maps for the Hopf algebras  $U_\sigma(\mathcal{S})$ ,  $U_\tau(\mathcal{S})$ ; these are straightforwardly obtained from the maps corresponding to their Hopf subalgebras  $U_z(h_4)$  and  $U_z(gl(2))$ . These nonlinear changes of basis allow us to write explicitly the Hopf Schrödinger algebras with classical commutators and non-cocommutative coproduct, so that the connection with the results of [29] can be established. The two quantum algebras  $U_\sigma(\mathcal{S})$ ,  $U_\tau(\mathcal{S})$  are analyzed separately in the sections 3 and 4, respectively. Finally, some remarks close the paper.

## 2 Twist maps for non-standard quantum algebras

### 2.1 Non-standard quantum $sl(2, \mathbb{R})$ algebra

The commutation rules and coproduct of the Hopf algebra  $U_z(sl(2, \mathbb{R}))$  in the original form deduced in [4] are given by

$$\begin{aligned} [H, X] &= 2 \frac{\sinh(zX)}{z} & [H, Y] &= -Y \cosh(zX) - \cosh(zX) Y & [X, Y] &= H \\ \Delta(X) &= 1 \otimes X + X \otimes 1 & \Delta(Y) &= e^{-zX} \otimes Y + Y \otimes e^{zX} \\ \Delta(H) &= e^{-zX} \otimes H + H \otimes e^{zX}. \end{aligned} \quad (2.1)$$

The nonlinear invertible map defined by [31]

$$J_+ = X \quad J_3 = e^{zX} H \quad J_- = e^{zX} (Y - z \sinh(zX)/4) \quad (2.2)$$

allows us to write the former structure of  $U_z(sl(2, \mathbb{R}))$  as follows:

$$\begin{aligned} [J_3, J_+] &= \frac{e^{2zJ_+} - 1}{z} & [J_3, J_-] &= -2J_- + zJ_3^2 & [J_+, J_-] &= J_3 \\ \Delta(J_+) &= 1 \otimes J_+ + J_+ \otimes 1 & \Delta(J_-) &= 1 \otimes J_- + J_- \otimes e^{2zJ_+} \\ \Delta(J_3) &= 1 \otimes J_3 + J_3 \otimes e^{2zJ_+}. \end{aligned} \quad (2.3)$$

We remark that in this basis the universal quantum  $R$ -matrix of  $U_z(sl(2, \mathbb{R}))$  adopts the factorized expression (1.3) and the classical  $r$ -matrix is (1.1).

Now if we apply to (2.3) a second invertible map given by

$$\mathcal{J}_+ = \frac{1 - e^{-2zJ_+}}{2z} \quad \mathcal{J}_3 = J_3 \quad \mathcal{J}_- = J_- - \frac{z}{2} J_3^2 \quad (2.4)$$

then we find the classical commutators of  $sl(2, \mathbb{R})$

$$[\mathcal{J}_3, \mathcal{J}_+] = 2\mathcal{J}_+ \quad [\mathcal{J}_3, \mathcal{J}_-] = -2\mathcal{J}_- \quad [\mathcal{J}_+, \mathcal{J}_-] = \mathcal{J}_3 \quad (2.5)$$

while the coproduct turns out to be

$$\begin{aligned} \Delta(\mathcal{J}_+) &= 1 \otimes \mathcal{J}_+ + \mathcal{J}_+ \otimes 1 - 2z\mathcal{J}_+ \otimes \mathcal{J}_+ \\ \Delta(\mathcal{J}_3) &= 1 \otimes \mathcal{J}_3 + \mathcal{J}_3 \otimes \frac{1}{1 - 2z\mathcal{J}_+} \\ \Delta(\mathcal{J}_-) &= 1 \otimes \mathcal{J}_- + \mathcal{J}_- \otimes \frac{1}{1 - 2z\mathcal{J}_+} - z\mathcal{J}_3 \otimes \frac{1}{1 - 2z\mathcal{J}_+} \mathcal{J}_3 \\ &\quad - z^2(\mathcal{J}_3^2 + 2\mathcal{J}_3) \otimes \frac{\mathcal{J}_+}{(1 - 2z\mathcal{J}_+)^2}. \end{aligned} \quad (2.6)$$

Note that although the new generator  $\mathcal{J}_+$  is non-primitive, its coproduct satisfies

$$\Delta((1 - 2z\mathcal{J}_+)^a) = (1 - 2z\mathcal{J}_+)^a \otimes (1 - 2z\mathcal{J}_+)^a \quad (2.7)$$

since the old generator  $J_+$  fulfils  $\Delta(e^{azJ_+}) = e^{azJ_+} \otimes e^{azJ_+}$  for any real number  $a$ .

The composition of both maps, (2.2) and (2.4), gives rise to

$$\mathcal{J}_+ = \frac{1 - T^{-2}}{2z} \quad \mathcal{J}_3 = TH \quad \mathcal{J}_- = TY - \frac{z}{2}(TH)^2 - \frac{z}{8}(T^2 - 1) \quad (2.8)$$

where  $T = e^{zX}$ . This (invertible) twist map carries the former Hopf structure of  $U_z(sl(2, \mathbb{R}))$  (2.1) to the last one characterized by (2.5) and (2.6) and is the so called minimal twist map obtained by Abdesselam *et al* [1].

We emphasize that the coproduct (2.6) has been explicitly obtained in a ‘closed’ form which is worth to be compared with the previous literature on nonlinear maps for the non-standard quantum  $sl(2, \mathbb{R})$  algebra [1, 32, 33, 34, 35] since, in general, the transformed coproduct has a very complicated form. In this respect see [34] where the corresponding map is used to construct the representation theory of  $U_z(sl(2, \mathbb{R}))$  and also [35] where the Clebsch–Gordan coefficients are computed.

## 2.2 Non-standard quantum $gl(2)$ algebra

A non-standard quantum deformation of  $gl(2)$  whose underlying Lie bialgebra is again generated by the classical  $r$ -matrix (1.1) was constructed in [9]. The Hopf algebra  $U_z(gl(2))$  reads

$$\begin{aligned} [J_3, J_+] &= \frac{e^{2zJ_+} - 1}{z} & [J_3, J_-] &= -2J_- + zJ_3^2 \\ [J_+, J_-] &= J_3 - I e^{2zJ_+} & [I, \cdot] &= 0 \end{aligned} \quad (2.9)$$

$$\begin{aligned} \Delta(J_+) &= 1 \otimes J_+ + J_+ \otimes 1 & \Delta(J_3) &= 1 \otimes J_3 + J_3 \otimes e^{2zJ_+} \\ \Delta(I) &= 1 \otimes I + I \otimes 1 & \Delta(J_-) &= 1 \otimes J_- + J_- \otimes e^{2zJ_+} + zJ_3 \otimes I e^{2zJ_+}. \end{aligned} \quad (2.10)$$

The universal quantum  $R$ -matrix of  $U_z(gl(2))$  is also given by (1.3).

The twist map which turns (2.9) into classical commutation rules is exactly the same as for  $U_z(sl(2, \mathbb{R}))$  given in (2.4) together with  $\mathcal{I} = I$ . The resulting Hopf structure is

$$[\mathcal{J}_3, \mathcal{J}_+] = 2\mathcal{J}_+ \quad [\mathcal{J}_3, \mathcal{J}_-] = -2\mathcal{J}_- \quad [\mathcal{J}_+, \mathcal{J}_-] = \mathcal{J}_3 - \mathcal{I} \quad [\mathcal{I}, \cdot] = 0 \quad (2.11)$$

$$\begin{aligned} \Delta(\mathcal{J}_+) &= 1 \otimes \mathcal{J}_+ + \mathcal{J}_+ \otimes 1 - 2z\mathcal{J}_+ \otimes \mathcal{J}_+ \\ \Delta(\mathcal{J}_3) &= 1 \otimes \mathcal{J}_3 + \mathcal{J}_3 \otimes \frac{1}{1 - 2z\mathcal{J}_+} \\ \Delta(\mathcal{J}_-) &= 1 \otimes \mathcal{J}_- + \mathcal{J}_- \otimes \frac{1}{1 - 2z\mathcal{J}_+} - z\mathcal{J}_3 \otimes \frac{1}{1 - 2z\mathcal{J}_+} (\mathcal{J}_3 - \mathcal{I}) \\ &\quad - z^2 (\mathcal{J}_3^2 + 2\mathcal{J}_3) \otimes \frac{\mathcal{J}_+}{(1 - 2z\mathcal{J}_+)^2} \\ \Delta(\mathcal{I}) &= 1 \otimes \mathcal{I} + \mathcal{I} \otimes 1. \end{aligned} \quad (2.12)$$

It is clear that  $U_z(gl(2))$  can be seen as an extended  $U_z(sl(2, \mathbb{R}))$  by  $\mathcal{I}$  which is a central and primitive generator; if we take  $\mathcal{I} = 0$  we recover the results of the above subsection.

In relation with this construction we recall that a two-parameter quantum  $gl(2)$  algebra,  $U_{g,h}(gl(2))$ , was introduced in [36]; it includes  $U_z(gl(2))$  as a particular case (when both deformation parameters are identified with  $z$ ). The Drinfeld twist operator and map for  $U_{g,h}(gl(2))$  were analyzed in [37].

### 2.3 Non-standard quantum Poincaré algebra

The Hopf algebra  $U_z(sl(2, \mathbb{R}))$  can be contracted to the non-standard (1+1) Poincaré algebra by means of the following transformation of the generators and deformation parameter [9]:

$$P_+ = \varepsilon J_+ \quad P_- = \varepsilon J_- \quad K = \frac{1}{2} J_3 \quad z \rightarrow 2 \frac{z}{\varepsilon}. \quad (2.13)$$

The limit  $\varepsilon \rightarrow 0$  gives rise to the Hopf algebra  $U_z(\mathcal{P})$  (written in a null-plane basis):

$$\begin{aligned} [K, P_+] &= \frac{e^{zP_+} - 1}{z} & [K, P_-] &= -P_- & [P_+, P_-] &= 0 \\ \Delta(P_+) &= 1 \otimes \tilde{P}_+ + P_+ \otimes 1 & \Delta(P_-) &= 1 \otimes P_- + P_- \otimes e^{zP_+} \\ \Delta(K) &= 1 \otimes K + K \otimes e^{zP_+} \end{aligned} \quad (2.14)$$

and the corresponding classical  $r$ -matrix is  $r = zK \wedge P_+$  while the universal  $R$ -matrix is similar to (1.3).

The contraction (2.13) allows us to obtain straightforwardly the twist map for  $U_z(\mathcal{P})$  and its resulting Hopf algebra; they are

$$\mathcal{P}_+ = \frac{1 - e^{-zP_+}}{z} \quad \mathcal{K} = K \quad \mathcal{P}_- = P_- \quad (2.15)$$

$$[\mathcal{K}, \mathcal{P}_+] = \mathcal{P}_+ \quad [\mathcal{K}, \mathcal{P}_-] = -\mathcal{P}_- \quad [\mathcal{P}_+, \mathcal{P}_-] = 0 \quad (2.16)$$

$$\begin{aligned} \Delta(\mathcal{P}_+) &= 1 \otimes \mathcal{P}_+ + \mathcal{P}_+ \otimes \frac{1 - z\mathcal{P}_+}{1 - z\mathcal{P}_+} \\ \Delta(\mathcal{K}) &= 1 \otimes \mathcal{K} + \mathcal{K} \otimes \frac{1}{1 - z\mathcal{P}_+} \\ \Delta(\mathcal{P}_-) &= 1 \otimes \mathcal{P}_- + \mathcal{P}_- \otimes \frac{1}{1 - z\mathcal{P}_+}. \end{aligned} \quad (2.17)$$

### 2.4 Non-standard quantum harmonic oscillator algebra

The Jordanian quantum oscillator algebra  $U_z(h_4)$  can be obtained by applying the following contraction [9] to the Hopf algebra  $U_z(gl(2))$  given in (2.9) and (2.10):

$$A_+ = \varepsilon J_+ \quad A_- = \varepsilon J_- \quad N = \frac{1}{2} J_3 \quad M = \varepsilon^2 I \quad z \rightarrow 2 \frac{z}{\varepsilon} \quad (2.18)$$

together with the limit  $\varepsilon \rightarrow 0$ . Thus we find the Hopf structure of  $U_z(h_4)$ :

$$\begin{aligned} [N, A_+] &= \frac{e^{zA_+} - 1}{z} & [N, A_-] &= -A_- & [A_-, A_+] &= Me^{zA_+} & [M, \cdot] &= 0 \\ \Delta(A_+) &= 1 \otimes A_+ + A_+ \otimes 1 & \Delta(N) &= 1 \otimes N + N \otimes e^{zA_+} \\ \Delta(M) &= 1 \otimes M + M \otimes 1 & \Delta(A_-) &= 1 \otimes A_- + A_- \otimes e^{zA_+} + zN \otimes Me^{zA_+} \end{aligned} \quad (2.19)$$

whose classical  $r$ -matrix is  $r = zN \wedge A_+$ .

The above contraction gives also the twist map and the corresponding Hopf algebra:

$$\mathcal{A}_+ = \frac{1 - e^{-zA_+}}{z} \quad \mathcal{N} = N \quad \mathcal{A}_- = A_- \quad \mathcal{M} = M \quad (2.20)$$

$$[\mathcal{N}, \mathcal{A}_+] = \mathcal{A}_+ \quad [\mathcal{N}, \mathcal{A}_-] = -\mathcal{A}_- \quad [\mathcal{A}_-, \mathcal{A}_+] = \mathcal{M} \quad [\mathcal{M}, \cdot] = 0 \quad (2.21)$$

$$\begin{aligned} \Delta(\mathcal{A}_+) &= 1 \otimes \mathcal{A}_+ + \mathcal{A}_+ \otimes 1 - z\mathcal{A}_+ \otimes \mathcal{A}_+ \\ \Delta(\mathcal{N}) &= 1 \otimes \mathcal{N} + \mathcal{N} \otimes \frac{1}{1 - z\mathcal{A}_+} \\ \Delta(\mathcal{A}_-) &= 1 \otimes \mathcal{A}_- + \mathcal{A}_- \otimes \frac{1}{1 - z\mathcal{A}_+} + z\mathcal{N} \otimes \frac{\mathcal{M}}{1 - z\mathcal{A}_+} \\ \Delta(\mathcal{M}) &= 1 \otimes \mathcal{M} + \mathcal{M} \otimes 1. \end{aligned} \quad (2.22)$$

If we set the central generator  $\mathcal{M} = 0$  and denote  $\mathcal{N} = \mathcal{K}$ ,  $\mathcal{A}_+ = \mathcal{P}_+$  and  $\mathcal{A}_- = \mathcal{P}_-$ , then we find the results concerning the non-standard (1+1) Poincaré algebra of section 2.3. Therefore we have completed the first part of the diagram of the Introduction at the level of (minimal) twist maps.

### 3 Discrete space Schrödinger symmetries

Let us consider the discrete version of the SE on a two-dimensional uniform lattice introduced in [29]

$$(\Delta_x^2 - 2m\Delta_t)\phi(x, t) = 0 \quad (3.1)$$

where the difference operators  $\Delta_x$  and  $\Delta_t$  can be expressed in terms of shift operators  $T_x = e^{\sigma\partial_x}$  and  $T_t = e^{\tau\partial_t}$  as

$$\Delta_x = \frac{T_x - 1}{\sigma} \quad \Delta_t = \frac{T_t - 1}{\tau}. \quad (3.2)$$

The parameters  $\sigma$  and  $\tau$  are the lattice constants in the space  $x$  and time  $t$  directions, respectively. The action of  $\Delta_x$  (resp.  $\Delta_t$ ) on a function  $\phi(x, t)$  consists in a discrete derivative, which in the limit  $\sigma \rightarrow 0$  (resp.  $\tau \rightarrow 0$ ) comes into  $\partial_x$  (resp.  $\partial_t$ ). We shall say that an operator  $\mathcal{O}$  is a symmetry of the linear equation  $E\phi(x, t) = 0$  if  $\mathcal{O}$  transforms solutions into solutions, that is, if  $\mathcal{O}$  is such that

$$E\mathcal{O} = \Lambda E \quad (3.3)$$

where  $\Lambda$  is another operator. In this way, the symmetries of the equation (3.1) were computed in [29] showing that they spanned the Schrödinger Lie algebra  $\mathcal{S}$ , which is exactly the same result as for the continuous case [18, 19].

### 3.1 Quantum Schrödinger algebra $U_\sigma(\mathcal{S})$ and discrete space Schrödinger equation

We consider the Schrödinger generators of time translation  $H$ , space translation  $P$ , Galilean boost  $K$ , dilation  $D$ , conformal transformation  $C$ , and central generator  $M$  [18, 19]. Let  $U_\sigma(\mathcal{S})$  be the quantum Schrödinger algebra [21] whose underlying Lie bialgebra is generated by the non-standard classical  $r$ -matrix

$$r = \sigma D' \wedge P \quad (3.4)$$

where  $D' = D + \frac{1}{2}M$ ; hereafter we shall use this notation in order to simplify some expressions. The coproduct of  $U_\sigma(\mathcal{S})$  has two primitive generators: the central one  $M$  and the space translation  $P$ ; it reads [21]

$$\begin{aligned} \Delta(M) &= 1 \otimes M + M \otimes 1 & \Delta(P) &= 1 \otimes P + P \otimes 1 \\ \Delta(H) &= 1 \otimes H + H \otimes e^{2\sigma P} \\ \Delta(K) &= 1 \otimes K + K \otimes e^{-\sigma P} + \sigma D' \otimes e^{-\sigma P} M \\ \Delta(D) &= 1 \otimes D + D \otimes e^{-\sigma P} + \frac{1}{2}M \otimes (e^{-\sigma P} - 1) \\ \Delta(C) &= 1 \otimes C + C \otimes e^{-2\sigma P} - \frac{\sigma}{2}K \otimes e^{-\sigma P} D' + \frac{\sigma}{2}D' \otimes e^{-\sigma P} (K - \sigma D' M) \end{aligned} \quad (3.5)$$

while the deformed commutation rules are given by

$$\begin{aligned} [D, P] &= \frac{1}{\sigma}(e^{-\sigma P} - 1) & [D, K] &= K & [K, P] &= M e^{-\sigma P} & [M, \cdot] &= 0 \\ [D, H] &= -2H & [D, C] &= 2C + \frac{\sigma}{2}K D' & [H, P] &= 0 \\ [H, C] &= \frac{1}{2}(1 + e^{\sigma P})D' - \frac{1}{2}M - \sigma K H & [K, C] &= \frac{\sigma}{2}K^2 \\ [P, C] &= -\frac{1}{2}(1 + e^{-\sigma P})K + \frac{\sigma}{2}e^{-\sigma P}M D' & [K, H] &= \frac{1}{\sigma}(e^{\sigma P} - 1). \end{aligned} \quad (3.6)$$

The relationship between  $U_\sigma(\mathcal{S})$  and a space discretization of the SE can be established by means of the following differential-difference realization of (3.6) in terms of the space and time coordinates  $(x, t)$ :

$$\begin{aligned} P &= \partial_x & H &= \partial_t & M &= m \\ K &= -t \left( \frac{e^{\sigma \partial_x} - 1}{\sigma} \right) - m x e^{-\sigma \partial_x} & D &= 2t \partial_t + x \left( \frac{1 - e^{-\sigma \partial_x}}{\sigma} \right) + \frac{1}{2} \\ C &= t^2 \partial_t e^{\sigma \partial_x} + t x \left( \frac{\sinh \sigma \partial_x}{\sigma} + \sigma m \partial_t e^{-\sigma \partial_x} \right) + \frac{1}{2} m x^2 e^{-\sigma \partial_x} \\ &\quad - \frac{1}{4} t \{ 1 - 3e^{\sigma \partial_x} + m(1 - e^{\sigma \partial_x}) \} - \frac{1}{4} \sigma m (1 - m) x e^{-\sigma \partial_x}. \end{aligned} \quad (3.7)$$

The limit  $\sigma \rightarrow 0$  gives the classical Schrödinger vector field representation. The Galilei generators  $\{K, H, P, M\}$  close a deformed subalgebra (but not a Hopf subalgebra) whose Casimir is

$$E_\sigma = \left( \frac{e^{\sigma P} - 1}{\sigma} \right)^2 - 2MH. \quad (3.8)$$



The action of  $E_\sigma$  on a function  $\phi(x, t)$  through (3.7) provides a space discretization of the SE by choosing for  $E_\sigma$  the zero eigenvalue:

$$E_\sigma \phi(x, t) = 0 \quad \implies \quad \left( \left( \frac{e^{\sigma \partial_x} - 1}{\sigma} \right)^2 - 2m \partial_t \right) \phi(x, t) = 0. \quad (3.9)$$

Furthermore, according to the definition of a symmetry operator (3.3) we find that the quantum algebra  $U_\sigma(\mathcal{S})$  is a symmetry algebra of (3.9) since their operators (3.7) verify

$$\begin{aligned} [E_\sigma, X] &= 0 \quad \text{for } X \in \{K, H, P, M\} & [E_\sigma, D] &= 2E_\sigma \\ [E_\sigma, C] &= \{t(e^{\sigma \partial_x} + 1) + \sigma m x e^{-\sigma \partial_x}\} E_\sigma. \end{aligned} \quad (3.10)$$

### 3.2 Twist map for $U_\sigma(\mathcal{S})$

The quantum  $h_4$  algebra described in the section 2.4 arises as a Hopf subalgebra of  $U_\sigma(\mathcal{S})$  once we rename the generators and deformation parameter of  $U_z(h_4)$  as

$$A_+ = P \quad A_- = K \quad N = -D - \frac{1}{2}M \equiv -D' \quad z = -\sigma \quad (3.11)$$

keeping  $M$  as the same central generator. Notice that under this identification the classical  $r$ -matrices of both quantum algebras coincide:  $r = zN \wedge A_+ = \sigma D' \wedge P$ . The embedding  $U_z(h_4) \subset U_\sigma(\mathcal{S})$  allows us to deduce straightforwardly the (minimal) twist map for  $U_\sigma(\mathcal{S})$ . The map associated to  $U_z(h_4)$  (2.20) written in the Schrödinger basis (3.11) reads

$$\mathcal{P} = \frac{e^{\sigma P} - 1}{\sigma} \quad \mathcal{D} = D \quad \mathcal{K} = K \quad \mathcal{M} = M. \quad (3.12)$$

The change of basis for  $U_\sigma(\mathcal{S})$  is completed with the transformation of the two remaining generators that turns out to be

$$\mathcal{H} = H \quad \mathcal{C} = C + \frac{\sigma}{2} K D'. \quad (3.13)$$

In this new basis the commutation rules (3.6) of the Hopf algebra  $U_\sigma(\mathcal{S})$  come into the Schrödinger Lie algebra:

$$\begin{aligned} [\mathcal{D}, \mathcal{P}] &= -\mathcal{P} & [\mathcal{D}, \mathcal{K}] &= \mathcal{K} & [\mathcal{K}, \mathcal{P}] &= \mathcal{M} & [\mathcal{M}, \cdot] &= 0 \\ [\mathcal{D}, \mathcal{H}] &= -2\mathcal{H} & [\mathcal{D}, \mathcal{C}] &= 2\mathcal{C} & [\mathcal{H}, \mathcal{C}] &= \mathcal{D} & [\mathcal{H}, \mathcal{P}] &= 0 \\ [\mathcal{P}, \mathcal{C}] &= -\mathcal{K} & [\mathcal{K}, \mathcal{H}] &= \mathcal{P} & [\mathcal{K}, \mathcal{C}] &= 0 \end{aligned} \quad (3.14)$$

and the coproduct is now given by

$$\begin{aligned} \Delta(\mathcal{M}) &= 1 \otimes \mathcal{M} + \mathcal{M} \otimes 1 \\ \Delta(\mathcal{P}) &= 1 \otimes \mathcal{P} + \mathcal{P} \otimes 1 + \sigma \mathcal{P} \otimes \mathcal{P} \\ \Delta(\mathcal{H}) &= 1 \otimes \mathcal{H} + \mathcal{H} \otimes (1 + \sigma \mathcal{P})^2 \\ \Delta(\mathcal{K}) &= 1 \otimes \mathcal{K} + \mathcal{K} \otimes \frac{1}{1 + \sigma \mathcal{P}} + \sigma \mathcal{D}' \otimes \frac{\mathcal{M}}{1 + \sigma \mathcal{P}} \end{aligned}$$

$$\begin{aligned}
\Delta(\mathcal{D}) &= 1 \otimes \mathcal{D} + \mathcal{D} \otimes \frac{1}{1 + \sigma \mathcal{P}} - \frac{1}{2} \mathcal{M} \otimes \frac{\sigma \mathcal{P}}{1 + \sigma \mathcal{P}} \\
\Delta(\mathcal{C}) &= 1 \otimes \mathcal{C} + \mathcal{C} \otimes \frac{1}{(1 + \sigma \mathcal{P})^2} + \sigma \mathcal{D}' \otimes \frac{1}{1 + \sigma \mathcal{P}} \mathcal{K} \\
&\quad + \frac{\sigma^2}{2} \mathcal{D}'(\mathcal{D}' - 1) \otimes \frac{\mathcal{M}}{(1 + \sigma \mathcal{P})^2}
\end{aligned} \tag{3.15}$$

where we have used again the shorthand notation  $\mathcal{D}' = \mathcal{D} + \frac{1}{2} \mathcal{M}$ . Note that the new generator  $\mathcal{P}$  is non-primitive but satisfies a property similar to (2.7).

The mapping defined by (3.12) and (3.13) transforms the differential-difference realization (3.7) into

$$\begin{aligned}
\mathcal{P} &= \Delta_x & \mathcal{H} &= \partial_t & \mathcal{M} &= m \\
\mathcal{K} &= -t\Delta_x - mxT_x^{-1} & \mathcal{D} &= 2t\partial_t + x\Delta_x T_x^{-1} + \frac{1}{2} \\
\mathcal{C} &= t^2\partial_t + tx\Delta_x T_x^{-1} + \frac{1}{2}m(x^2 - \sigma x)T_x^{-2} + \frac{1}{2}t
\end{aligned} \tag{3.16}$$

where  $\Delta_x$  and  $T_x$  are the difference and shift operators defined by (3.2). Obviously, the Casimir of the Galilei subalgebra  $E_\sigma$  (3.8) leads to the classical one

$$E = \mathcal{P}^2 - 2\mathcal{M}\mathcal{H} \tag{3.17}$$

so that the discretized SE obtained as the realization (3.16) of  $E\phi(x, t) = 0$  is

$$(\Delta_x^2 - 2m\partial_t)\phi(x, t) = 0 \tag{3.18}$$

which coincides with (3.9); the operators (3.16) are symmetries of this equation satisfying

$$[E, X] = 0 \quad X \in \{\mathcal{K}, \mathcal{H}, \mathcal{P}, \mathcal{M}\} \quad [E, \mathcal{D}] = 2E \quad [E, \mathcal{C}] = 2tE. \tag{3.19}$$

### 3.3 Relation of $U_\sigma(\mathcal{S})$ with the Lie symmetry approach

The discrete space SE (3.18) is just the limit  $\tau \rightarrow 0$  of the equation (3.1) considered in [29]. Hence it is rather natural to expect a connection between the differential-difference symmetries obtained in [29] which close the Lie Schrödinger algebra (3.14) and our realization of  $U_\sigma(\mathcal{S})$ . Although the operators (3.16) do not coincide with those given in [29] we will show that indeed both realizations are related by means of a similarity transformation (see [1] for  $U_z(sl(2, \mathbb{R}))$ ).

The twist map defined by

$$\begin{aligned}
\mathcal{P} &= \frac{e^{\sigma P} - 1}{\sigma} & \mathcal{D} &= D + \frac{1}{2}(1 - e^{-\sigma P}) & \mathcal{K} &= K - \frac{\sigma}{2} M e^{-\sigma P} \\
\mathcal{M} &= M & \mathcal{H} &= H & \mathcal{C} &= C + \frac{\sigma}{2} K D' - \frac{\sigma}{2} K e^{-\sigma P} - \frac{\sigma^2}{8} M e^{-2\sigma P}
\end{aligned} \tag{3.20}$$

is equivalent to the one defined by (3.12) and (3.13) since it gives rise to the *same* Schrödinger Lie algebra (3.14) and non-cocommutative coproduct (3.15). However, the new map applied to the realization (3.7) leads to

$$\begin{aligned}\mathcal{P} &= \Delta_x & \mathcal{H} &= \partial_t & \mathcal{M} &= m \\ \mathcal{K} &= -t\Delta_x - mxT_x^{-1} - \frac{m\sigma}{2}T_x^{-1} & \mathcal{D} &= 2t\partial_t + x\Delta_x T_x^{-1} - \frac{1}{2}T_x^{-1} + 1 \\ \mathcal{C} &= t^2\partial_t + tx\Delta_x T_x^{-1} + \frac{1}{2}mx^2T_x^{-2} + t\left(1 - \frac{1}{2}T_x^{-1}\right) - \frac{m\sigma^2}{8}T_x^{-2}\end{aligned}\quad (3.21)$$

which are just the symmetry operators of the equation (3.1) obtained in [29], provided the continuous time limit  $\tau \rightarrow 0$  is performed,  $m = \frac{1}{2}$  and  $\mathcal{K} \rightarrow -2\mathcal{K}$ . In other words, we have shown that the space differential-difference SE introduced in [29] has  $U_\sigma(\mathcal{S})$  as its Hopf symmetry algebra; the operators (3.21) fulfil the same relations (3.19). The deformation parameter  $\sigma$  is interpreted as the lattice step in the  $x$  coordinate, meanwhile the time  $t$  remains a continuous variable. We also remark that, by using (3.21), the solutions of (3.18) have been obtained in [29] for  $m = \frac{1}{2}$ .

## 4 Discrete time Schrödinger symmetries

### 4.1 Quantum Schrödinger algebra $U_\tau(\mathcal{S})$ and discrete time Schrödinger equation

A similar procedure can be applied to the quantum Schrödinger algebra  $U_\tau(\mathcal{S})$  [20] coming from the non-standard classical  $r$ -matrix

$$r = \frac{\tau}{2}D' \wedge H. \quad (4.1)$$

The coproduct of  $U_\tau(\mathcal{S})$  has two primitive generators: the central one  $M$  and the time translation  $H$  (instead of  $P$ ); it is given by [20]

$$\begin{aligned}\Delta(M) &= 1 \otimes M + M \otimes 1 & \Delta(H) &= 1 \otimes H + H \otimes 1 \\ \Delta(P) &= 1 \otimes P + P \otimes e^{\tau H/2} \\ \Delta(K) &= 1 \otimes K + K \otimes e^{-\tau H/2} + \frac{\tau}{2}D' \otimes e^{-\tau H}P \\ \Delta(D) &= 1 \otimes D + D \otimes e^{-\tau H} + \frac{1}{2}M \otimes (e^{-\tau H} - 1) \\ \Delta(C) &= 1 \otimes C + C \otimes e^{-\tau H} + \frac{\tau}{4}D' \otimes e^{-\tau H}M\end{aligned}\quad (4.2)$$

and the compatible commutation rules are

$$\begin{aligned}[D, P] &= -P & [D, K] &= K & [K, P] &= M & [M, \cdot] &= 0 \\ [D, H] &= \frac{2}{\tau}(e^{-\tau H} - 1) & [D, C] &= 2C - \frac{\tau}{2}(D')^2 & [H, P] &= 0 \\ [H, C] &= D' - \frac{1}{2}Me^{-\tau H} & [K, C] &= -\frac{\tau}{4}(D'K + KD') \\ [P, C] &= -K + \frac{\tau}{4}(D'P + PD') & [K, H] &= e^{-\tau H}P.\end{aligned}\quad (4.3)$$

A differential-difference realization of (4.3) reads [20]

$$\begin{aligned}
H &= \partial_t & P &= \partial_x & M &= m \\
K &= -(t - \tau)e^{-\tau\partial_t}\partial_x - mx & D &= 2(t - \tau)\left(\frac{1 - e^{-\tau\partial_t}}{\tau}\right) + x\partial_x + \frac{1}{2} \\
C &= (t^2 + \tau bt)\left(\frac{1 - e^{-\tau\partial_t}}{\tau}\right) + tx\partial_x + \frac{1}{2}t + \frac{1}{2}mx^2 + \tau(b + 1)e^{-\tau\partial_t} \\
&\quad + \frac{\tau}{4}x^2\partial_x^2 + \frac{\tau}{2}(b + 1)x\partial_x + \frac{\tau}{4}(b + 1/2)^2
\end{aligned} \tag{4.4}$$

where  $b = \frac{m}{2} - 2$ . A time discretization of the SE is obtained by considering the deformed Casimir of the Galilei subalgebra

$$E_\tau = P^2 - 2M\left(\frac{e^{\tau H} - 1}{\tau}\right) \tag{4.5}$$

written in terms of the realization (4.4):

$$E_\tau\phi(x, t) = 0 \implies \left(\partial_x^2 - 2m\left(\frac{e^{\tau\partial_t} - 1}{\tau}\right)\right)\phi(x, t) = 0. \tag{4.6}$$

Under the realization (4.4) the generators of  $U_\tau(\mathcal{S})$  are symmetry operators of this equation as they satisfy

$$\begin{aligned}
[E_\tau, X] &= 0 \quad X \in \{K, H, P, M\} & [E_\tau, D] &= 2E_\tau \\
[E_\tau, C] &= 2\left\{t - \frac{\tau}{4}(1 - m - 2x\partial_x)\right\}E_\tau.
\end{aligned} \tag{4.7}$$

## 4.2 Twist map for $U_\tau(\mathcal{S})$

The quantum  $gl(2)$  algebra studied in the section 2.2 arises as a Hopf subalgebra of  $U_\tau(\mathcal{S})$  under the following identification:

$$J_+ = H \quad J_- = -C \quad J_3 = -D - \frac{1}{2}M \equiv -D' \quad I = -\frac{1}{2}M \quad z = -\frac{1}{2}\tau. \tag{4.8}$$

In the Schrödinger basis the twist map for  $U_z(gl(2))$  (2.4) (which is the same as for  $U_z(sl(2, \mathbb{R}))$ ) is given by

$$\mathcal{H} = \frac{e^{\tau H} - 1}{\tau} \quad \mathcal{D} = D \quad \mathcal{C} = C - \frac{\tau}{4}(D')^2 \quad \mathcal{M} = M. \tag{4.9}$$

The twist map for  $U_\tau(\mathcal{S}) \supset U_z(gl(2))$  is completed with the transformation of the two generators out of  $U_z(gl(2))$  which is simply

$$\mathcal{P} = P \quad \mathcal{K} = K. \tag{4.10}$$

If we apply (4.9) and (4.10) to (4.2) and (4.3), we find again the classical commutation rules of the Schrödinger algebra (3.14) while the coproduct reads now

$$\Delta(\mathcal{M}) = 1 \otimes \mathcal{M} + \mathcal{M} \otimes 1$$

$$\begin{aligned}
\Delta(\mathcal{H}) &= 1 \otimes \mathcal{H} + \mathcal{H} \otimes 1 + \tau \mathcal{H} \otimes \mathcal{H} \\
\Delta(\mathcal{P}) &= 1 \otimes \mathcal{P} + \mathcal{P} \otimes (1 + \tau \mathcal{H})^{1/2} \\
\Delta(\mathcal{K}) &= 1 \otimes \mathcal{K} + \mathcal{K} \otimes \frac{1}{(1 + \tau \mathcal{H})^{1/2}} + \frac{\tau}{2} \mathcal{D}' \otimes \frac{\mathcal{P}}{1 + \tau \mathcal{H}} \\
\Delta(\mathcal{D}) &= 1 \otimes \mathcal{D} + \mathcal{D} \otimes \frac{1}{1 + \tau \mathcal{H}} - \frac{1}{2} \mathcal{M} \otimes \frac{\tau \mathcal{H}}{1 + \tau \mathcal{H}} \\
\Delta(\mathcal{C}) &= 1 \otimes \mathcal{C} + \mathcal{C} \otimes \frac{1}{1 + \tau \mathcal{H}} - \frac{\tau}{2} \mathcal{D}' \otimes \frac{1}{1 + \tau \mathcal{H}} \mathcal{D} \\
&\quad + \frac{\tau}{4} \mathcal{D}' (\mathcal{D}' - 2) \otimes \frac{\tau \mathcal{H}}{(1 + \tau \mathcal{H})^2}.
\end{aligned} \tag{4.11}$$

In this new basis the realization (4.4) turns out to be

$$\begin{aligned}
\mathcal{H} &= \Delta_t & \mathcal{P} &= \partial_x & \mathcal{M} &= m \\
\mathcal{K} &= -(t - \tau) \partial_x T_t^{-1} - mx & \mathcal{D} &= 2(t - \tau) \Delta_t T_t^{-1} + x \partial_x + \frac{1}{2} \\
\mathcal{C} &= t^2 \Delta_t T_t^{-2} + x(t - \tau) \partial_x T_t^{-1} + \frac{1}{2} m x^2 + 3t T_t^{-2} - \frac{5}{2} t T_t^{-1} - 2\tau T_t^{-2} + \frac{3\tau}{2} T_t^{-1}.
\end{aligned} \tag{4.12}$$

The corresponding discretized SE is provided by the (non-deformed) Casimir  $E$  of the Galilei subalgebra (3.17) written through (4.12) leading again to (4.6):

$$(\partial_x^2 - 2m \Delta_t) \phi(x, t) = 0. \tag{4.13}$$

The new operators (4.12) are symmetries of this equation satisfying

$$[E, X] = 0 \quad X \in \{\mathcal{K}, \mathcal{H}, \mathcal{P}, \mathcal{M}\} \quad [E, \mathcal{D}] = 2E \quad [E, \mathcal{C}] = 2(t - \tau) T_t^{-1} E. \tag{4.14}$$

### 4.3 Relation of $U_\tau(\mathcal{S})$ with the Lie symmetry approach

The connection with the time discretization of the SE analyzed in [29] is provided by the twist map defined by

$$\begin{aligned}
\mathcal{H} &= \frac{e^{\tau H} - 1}{\tau} & \mathcal{D} &= D + 2(1 - e^{-\tau H}) & \mathcal{C} &= C - \frac{\tau}{4} (D')^2 + \tau D e^{-\tau H} \\
\mathcal{M} &= M & \mathcal{P} &= P & \mathcal{K} &= K - \tau P e^{-\tau H}.
\end{aligned} \tag{4.15}$$

This nonlinear map is a similarity transformation of the former change of basis defined by (4.9) and (4.10) since it leads to the same Lie Schrödinger commutators (3.14) and non-cocommutative coproduct (4.11). Under this map the realization (4.4) becomes

$$\begin{aligned}
\mathcal{H} &= \Delta_t & \mathcal{P} &= \partial_x & \mathcal{M} &= m \\
\mathcal{K} &= -t \partial_x T_t^{-1} - mx & \mathcal{D} &= 2t \Delta_t T_t^{-1} + x \partial_x + \frac{1}{2} \\
\mathcal{C} &= t^2 \Delta_t T_t^{-2} + tx \partial_x T_t^{-1} + \frac{1}{2} m x^2 + t \left( T_t^{-2} - \frac{1}{2} T_t^{-1} \right).
\end{aligned} \tag{4.16}$$

These difference-differential operators are the limit  $\sigma \rightarrow 0$  of the symmetry operators obtained in [29] once we set  $m = \frac{1}{2}$  and  $\mathcal{K} \rightarrow -2\mathcal{K}$ . The corresponding discretized SE is again (4.13) and the new operators are symmetries of this equation satisfying

$$[E, X] = 0 \quad X \in \{\mathcal{K}, \mathcal{H}, \mathcal{P}, \mathcal{M}\} \quad [E, \mathcal{D}] = 2E \quad [E, \mathcal{C}] = 2tT_t^{-1}E. \quad (4.17)$$

Henceforth we have explicitly shown that the space discretization of the SE on a uniform lattice formerly studied in [29] within a pure Lie algebra approach has actually a quantum algebra symmetry associated to the Hopf algebra  $U_\tau(\mathcal{S})$ . Consequently the deformation parameter  $\tau$  is the time lattice step on this discrete time SE (the space coordinate  $x$  remains as a continuous variable). In this way, the relationships displayed in the r.h.s. of the diagram of the Introduction have been studied at the level of twist maps.

## 5 Concluding remarks

We have explicitly shown that the symmetry algebra [29] of the space discretization of the SE obtained from (3.1) by taking the limit  $\tau \rightarrow 0$  is just the quantum Schrödinger algebra  $U_\sigma(\mathcal{S})$  [21] and the deformation parameter  $\sigma$  is exactly the space lattice constant. Likewise, we have also shown that the time discretization of the SE obtained from (3.1) by means of the limit  $\sigma \rightarrow 0$  has the quantum Schrödinger algebra  $U_\tau(\mathcal{S})$  [20] as its symmetry algebra; in this case, the time lattice step  $\tau$  plays the role of the deformation parameter. Consequently, a direct relationship between non-standard (or Jordanian) deformations and regular lattice discretizations has been established.

We wish to point out that the existence of a Hopf algebra structure for the symmetries of a given equation associated to an elementary system allows us to write equations of composed systems keeping the same symmetry algebra [38, 39]. In order to use this property for the two discrete SE's here discussed, we see that only the last commutator in either (3.19) or (4.14) involving the conformal generator  $\mathcal{C}$  is not algebraic, but depends explicitly on the chosen representation (the same happens at the continuous level). Therefore the composed systems characterized by the equation  $\Delta(E)\phi = 0$  will have, by construction,  $\Delta(\mathcal{H}), \Delta(\mathcal{P}), \Delta(\mathcal{K}), \Delta(\mathcal{D})$ , and  $\Delta(\mathcal{M})$  as symmetry operators (moreover they close a Hopf subalgebra!). However, in general, this will not be the case for  $\Delta(\mathcal{C})$ , and a further study on the behaviour of this operator is needed in order to construct coupled equations with full quantum Schrödinger algebra symmetry.

Finally, we stress that the applicability of the constructive approach presented here is not limited to the cases analyzed before, since it could be directly extended to other quantum algebras by means of their corresponding differential-difference realizations. In particular, the results of this paper indicate that there should exist an analogous relationship between the discrete symmetries of the  $(1+1)$  wave equation on a uniform lattice obtained in [30] and some non-standard quantum deformation of the algebra  $so(2,2)$ . Work on this line is in progress.

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